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SOME ONE-RELATOR HOPFIAN GROUPS

BY

DONALD J. COLLINS

ABSTRACT. The group presented by

$$(a, t; t^{-1}a^{l}t = a^{m})$$

is non-Hopfian if $l, m \neq \pm 1$ and $\pi(l) \neq \pi(m)$, where $\pi(l)$ and $\pi(m)$ denote the sets of prime divisors of l and m. By contrast, we prove that if w is a word of the free group $F(a_1, a_2)$ which is not primitive and not a proper power, then the group

$$(a_1, a_2, t; t^{-1}w^{l}t = w^{m})$$

is Hopfian.

A group is Hopfian if every surjective endomorphism is an automorphism. The best-known example of a non-Hopfian group is probably the group

$$G = (a, t; t^{-1}a^{2}t = a^{3})$$

given by G. Baumslag and D. Solitar in [2]. In fact, Baumslag and Solitar considered all groups

$$G(l, m) = (a, t; t^{-1}a^{l}t = a^{m})$$

and we may summarise their conclusions as follows (incorporating a correction due to S. Meskin [7]):

If $|l| = 1$ or $|m| = 1$ or $|l| = |m|$ then $G(l, m)$ is residually finite and hence Hopfian.

Otherwise $G(l, m)$ is Hopfian if and only if $\pi(l) = \pi(m)$ (where $\pi(l)$, $\pi(m)$ denote the sets of prime divisors of l and m respectively).

These results show the delicacy of the Hopfian property and this is further illustrated by the theorem of G. Baumslag in [1] asserting that the group

$$G(l, m, n) = \left(a, t; (t^{-1}a^{l}ta^{-m})^{n} = 1\right)$$

is residually finite when $n > 1$ and l and m are coprime. In the same paper, Baumslag also states that if the one-relator group $H(1) = (a_1, a_2, a_3, \ldots;$ $r = 1)$, where r is not a proper power, is Hopfian, then so too is the one-relator group

$$H(n) = (a_1, a_2, a_3, \ldots; r^{n} = 1)$$

Hopfian.

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We seek to investigate groups of the form

$$G = (a_1, a_2, \ldots, a_n, t; t^{-1}w^l t = w^m)$$

where w is a word in the free group F on $a_1, a_2, \ldots, a_n$–and w is not a proper power in F. Meskin proved in [7] that if G is residually finite then $|l| = 1$ or $|m| = 1$ or $|l| = |m|$ and G. Baumslag (unpublished) has established the converse of this. In view of Baumslag's result, in determining whether or not G, as given above, is Hopfian we need only consider the situation in which $|l| \neq 1$, $|m| \neq 1$ and $|l| \neq |m|$.

If w is a primitive element of F then our given G is isomorphic to the free product of the Baumslag-Solitar group $G(l, m)$ and a free group of rank $(n - 1)$. In this situation G is Hopfian if and only if $G(l, m)$ is Hopfian. For I. M. S. Dey and H. Neumann proved in [4] that the free product of two finitely generated Hopfian groups is Hopfian–and it is easy to see that a free product is non-Hopfian if one of the factors is non-Hopfian.

Our contribution is the following theorem.

THEOREM. *Let $G = (a_1, a_2, t; t^{-1}w^l t = w^m)$ where w is a word in a_1, a_2 that is not primitive and not a proper power in the free group $F(a_1, a_2)$. Then G is Hopfian.*

At present we are unable to extend our results beyond the case $n = 2$.

We begin by summarising our notation. We write G for the group

$$(a_1, a_2, \ldots, a_n; t^{-1}w^l t = w^m)$$

where w is a word in $a_1, a_2, \ldots, a_n$ that is not primitive and not a proper power in the free group $F = F(a_1, a_2, \ldots, a_n)$. Eventually we shall put $n = 2$ but some of our results are valid for arbitrary n. When convenient we shall sometimes write $l[-1] = l$ and $l[1] = m$. As noted previously we may assume $|l| \neq |m|$–we shall not need to assume $|l| \neq 1$ and $|m| \neq 1$.

We write

$$K = (a_1, a_2, \ldots, a_n; w = 1) \quad \text{and} \quad L = K * \langle t \rangle$$

for the free product of K with an infinite cyclic group on t. There is a canonical epimorphism $\theta\colon G \to L$ obtained by putting $w = 1$.

We shall rely to a great extent on the theory of HNN-extensions and the corresponding normal form theorem and conjugacy lemma (see Chapter II of C. F. Miller III [8]–where HNN-extensions are called Britton extensions).

LEMMA 1. *G is an HNN-extension of F. In particular F is a subgroup of G.*

PROOF. This is obvious. Q.E.D.

We write $u \sim v$ to mean that u and v are conjugate elements of G. We write $u \sim F$ to mean that there exists $y \in F$ such that $u \sim y$. Given $u \in G$ we say that u is in *normal form* or is *t-reduced* if u does not contain a subword $t^{-\varepsilon}xt^{\varepsilon}$ with $x \in \langle w^{l[-\varepsilon]}\rangle$.

LEMMA 2. *Let $\phi: G \to G$ be an endomorphism.*

(i) *Then there is an inner automorphism ψ such that $w\phi\psi \in \langle w\rangle$. If $w\phi \in F$, then ψ corresponds to an element of F.*

(ii) *If $w\phi \in \langle w\rangle$ and $w\phi \neq 1$ then $t\phi = w^{\rho_0}t^{\varepsilon_1}w^{\rho_1}\cdots t_r^{\varepsilon}w^{\rho_r}$ where $\rho_i \in \mathbf{Z}$, $\varepsilon_i = \pm 1$ and $\sum_{i=1}^{r}\varepsilon_i = 1$.*

PROOF. (i) We know that $(w\phi)^l \sim (w\phi)^m$. Since $|l| \neq |m|$ the conjugacy lemma for HNN-extensions shows that $w\phi \sim F$. Without loss of generality we may assume $w\phi \in F$.

We have $(t\phi)^{-1}(w\phi)^l(t\phi) = (w\phi)^m$. Since $l \neq m$, $t\phi \not\in F$. Let us write

$$t = x_0t^{\varepsilon_1}x_1t^{\varepsilon_2}\cdots x_{r-1}t^{\varepsilon_r}x_r$$

where $\varepsilon_i = \pm 1$ and $x_i \in F$. Then $x_0^{-1}(w\phi)^lx_0 \in \langle w^{l[-\varepsilon_0]}\rangle$. Hence $\langle x_0^{-1}(w\phi)x_0\rangle \cap \langle w\rangle \neq 1$ giving $x_0^{-1}(w\phi)x_0 \in \langle w\rangle$, since F is free and w is not a proper power. This proves (i).

(ii) Let $w\phi = w^k$ so that $(t\phi)^{-1}w^{kl}(t\phi) = w^{km}$. The lemma follows from the fact that if u is any element of G and $u^{-1}w^pu = w^q$ then $u = w^{\rho_0}t^{\varepsilon_1}w^{\rho_1}\cdots t^{\varepsilon_r}w^{\rho_r}$ and $q = p(m/l)^{\sigma}$ where $\sigma = \sum_{i=1}^{r}\varepsilon_i$. This fact is established by induction on the number of occurrences of t in the normal form of u and relies heavily on the fact that if $x^{-1}w^rx = w^s$, where $x \in F$, then $r = s$ and $x \in \langle w\rangle$. (Again we do use the fact that w is not a proper power.)

LEMMA 3. *Let $u = t^{\varepsilon_1}w^{\rho_1}t^{\varepsilon_2}\cdots w^{\rho_{r-1}}t^{\varepsilon_r}$ be in normal form and suppose that*

(i) $u^{-1}w^lu = w^m$,

(ii) $G = \langle a, b, u\rangle$.

Then $u = t$, i.e. $r = 1$ and $\varepsilon_1 = 1$.

PROOF. We distinguish two cases.

Case 1. Suppose $l \nmid m$ and $m \nmid l$. Then $w^l \not\in \langle w^m\rangle$ and $w^m \not\in \langle w^l\rangle$. Since $u^{-1}w^lu = w^m$ we deduce that $\varepsilon_1 = 1 = \varepsilon_r$.

There must exist equalities of the form

$$t = y_0u^{\eta_1}y_1\cdots u^{\eta_s}y_s, \qquad y_i \in F, \eta_i = \pm 1, \tag{1}$$

since $G = \langle a, b, u\rangle$. Among all such we consider one with s minimal–we claim that then $s = 1$.

Suppose $s > 1$; then there must exist i such that in reducing $u^{\eta_{i-1}}y_{i-1}u^{\eta_i}y_iu^{\eta_{i+1}}$ to normal form the occurrences of t in u^{η_i} are eliminated. For if this were false the normal form of the right-hand side of (1) would contain at least s occurrences of t.

The minimality of s is contradicted immediately if $u^{\eta_{i-1}}y_{i-1}u^{\eta_i} \in F$ or

$u^{\eta_i}y_iu^{\eta_{i+1}} \in F$. So part of u^{η_i} is cancelled by $u^{\eta_{i-1}}$ and part by $u^{\eta_{i+1}}$ (strictly only the occurrences of t are cancelled). Since $\varepsilon_1 = \varepsilon_r$ we have $\eta_{i-1} + \eta_i = 0 = \eta_i + \eta_{i+1}$.

Suppose $\eta_{i-1} = 1$; then we may write $u = u_1u_2$ where $u_2y_{i-1}u_2^{-1} = x_{i-1} \in F$ and $u_1^{-1}y_iu_1 = x_i \in F$. Now in view of the form of u it is clear that both x_{i-1} and x_i lie in $\langle w \rangle$. In particular $x_{i-1}x_i = x_ix_{i-1}$. Then

$$uy_{i-1}u^{-1}y_iu = u_1x_{i-1}x_iu_2 = u_1x_ix_{i-1}u_2 = y_iuy_{i-1}.$$

Again the minimality is contradicted. The argument when $\eta_{i-1} = -1$ is similar.

So $t = y_0u^{\eta_1}y_1$; as u is in normal form we obtain $\eta_1 = 1$ and $r = 0$ as required.

Case 2. Suppose $l|m$ or $m|l$. Here there are two subcases but they can be treated similarly. So suppose that $m = lm_0$; then $m \nmid l$ as $|l| \neq |m|$.

From the fact that $u^{-1}w^lu = w^m$ we see that $\varepsilon_1 = 1$. However $w^m \in \langle w^l \rangle$ so we must allow for the possibility that $\varepsilon_r = -1$.

If in fact $\varepsilon_r = 1$ we can argue as in Case 1. So suppose $\varepsilon_r = -1$; we consider equations of the form

$$t = z_0u^{\lambda_1}z_1 \cdots u^{\lambda_s}z_s, \qquad z_i \in F, \lambda_i \in \mathbf{Z}, \lambda_i \neq 0.$$

As $G = \langle a, b, u \rangle$ such equations must exist. We again want to consider such an equation with s minimal.

We claim that in this event

(a) if $i > 1$ and $\lambda_i > 0$, then $z_{i-1} \notin \langle w^l \rangle$;

(b) if $i < s$ and $\lambda_i < 0$, then $z_i \notin \langle w^l \rangle$.

We note that for any $\lambda > 0$ and any $j \in \mathbf{Z}$, $u^{-\lambda}w^{jl}u^{\lambda} = w^{jlm_0^{\lambda}}$. Hence, if (a) is violated then for some $i > 1$

$$u^{\lambda_{i-1}}z_{i-1}u^{\lambda_i} = u^{\lambda_{i-1}+\lambda_i}z'_{i-1}, \quad \text{where } z'_{i-1} = u^{-\lambda_i}z_iu^{\lambda_i}.$$

Clearly the minimality of s is contradicted. Similarly if (b) is violated, then, for some $i < s$,

$$u^{\lambda_i}z_iu^{\lambda_{i+1}} = z'_iu^{\lambda_i+\lambda_{i+1}}$$

which again is contradictory.

Now write $u = u_1u_2u_1^{-1}$ where u_2 is cyclically t-reduced. Since $\varepsilon_1 = 1$ and $\varepsilon_r = -1$, $u_1 \notin F$. Also, since $u^{-1}w^lu = w^m$, $\Sigma_{i=1}^r\varepsilon_i = 1$ so that $u_2 \notin F$. Our equation becomes

$$t = z_0u_1u_2^{\lambda_1}u_1^{-1}z_1 \cdots u_1u_2^{\lambda_s}u_1^{-1}z_s.$$

Suppose $s > 1$; there cannot exist i such that $u_1u_2^{\lambda_i}u_1^{-1}z_i\, u_1u_2^{\lambda_{i+1}}u_1^{-1} \in F$. It follows that there exists i such that in reducing

$$u_1u_2^{\lambda_{i-1}}u_1^{-1}z_{i-1}u_1u_2^{\lambda_i}u_1^{-1}z_iu_1u_2^{\lambda_{i+1}}u_1^{-1}$$

to normal form, the occurrences of t in $u_1u_2^{\lambda}u_1^{-1}$ are eliminated partly from the left and partly from the right. This means that $z_{i-1} \in \langle w^l \rangle$ and $z_i \in \langle w^l \rangle$. The former implies that $\lambda_i < 0$ and the latter that $\lambda_i > 0$.

We conclude that $t = z_0u_1u_2^{\lambda_1}u_1^{-1}z_1$ which is impossible since the right-hand side is in normal form and has at least three occurrences of t. Q.E.D.

COROLLARY 4. *Let* ϕ: $G \to G$ *be a surjective endomorphism such that* $F\phi \subseteq F$ *and* $w\phi \in \langle w \rangle$, $w\phi \neq 1$. *Then* $t\phi = w^{\rho_0}tw^{\rho_1}$.

PROOF. By Lemma 2, $t\phi = w^{\rho_0}t^{\varepsilon_1}w^{\rho_1} \cdots t^{\varepsilon_r}w^{\rho_r}$. Let $u = t^{\varepsilon_1}w^{\rho_1} \cdots w^{\rho_{r-1}}t^{\varepsilon_r}$; since $G = \langle a\phi, b\phi, t\phi \rangle$ and $F\phi \subseteq F$ we obtain $G = \langle a, b, u \rangle$. By Lemma 3, $u = t$. Q.E.D.

PROPOSITION 5. *Let* $G = (a_1, a_2, \ldots, a_n, t;\ t^{-1}w^lt = w^m)$ *where* w *is not primitive and not a proper power in the free group* $F = F(a_1, a_2, \ldots, a_n)$. *If* ϕ *is a surjective endomorphism such that* $F\phi \subseteq F$, *then* ϕ *is an automorphism.*

PROOF. Certainly $w\phi \in F$ and, using Lemma 2, we may assume $w\phi \in \langle w \rangle$. More explicitly this asserts that for some $k \in \mathbf{Z}$

$$w(a_1\phi, a_2\phi, \ldots, a_n\phi) = w^k$$

and this equality holds in F. Suppose $k \neq \pm 1$; by a theorem of G. Baumslag and A. Steinberg in [3], the rank of the group $\langle a_1\phi, a_2\phi, \ldots, a_n\phi, w \rangle$ is at most $(n-1)$. Then certainly the rank of $\langle a_1\phi, a_2\phi, \ldots, a_n\phi \rangle$ is at most $(n-1)$ and so G can be generated by at most n elements. This means that

$$L = (a_1, a_2, \ldots, a_n;\ w = 1) * \langle t \rangle$$

can be generated by at most n elements. By Gruško's theorem

$$K = (a_1, a_2, \ldots, a_n;\ w = 1)$$

is generated by at most $(n-1)$ elements. By a theorem of W. Magnus (Corollary 5.14.2 of [5]), K is a free group. By Whitehead's theorem (Theorem N.3 of Chapter 3 of [5]), w is primitive in F. Thus $w\phi = w^{\pm 1}$.

Now by the same argument as above, the group $\langle a_1\phi, a_2\phi, \ldots, a_n\phi \rangle$ is free of rank n and thus ϕ is injective on F. Suppose $v\phi = 1$ and $v \neq 1$. Then certainly $v \notin F$. By the normal form theorem for HNN-extensions v must contain a subword $t^{-\varepsilon}zt^{\varepsilon}$ where $z \in \langle w^{l[-\varepsilon]} \rangle$. By Corollary 4, $t\phi = w^{\rho_0}tw^{\rho_1}$ and so v must contain a subword $t^{-\varepsilon}yt^{\varepsilon}$ such that $y\phi = z = w^{kl[-\varepsilon]}$. Now we know that $w\phi = w^{\eta}$, $\eta = \pm 1$ and hence $(w^{\eta kl[-\varepsilon]})\phi = y\phi$. As ϕ is injective on F, $y \in \langle w^{l[-\varepsilon]} \rangle$. Inductively, this means that $v \in F$ which is impossible. Q.E.D.

We have on the face of it proved a little more than was necessary to obtain Proposition 5. It would suffice to prove that $t\phi$ was of the form $w^{\rho_0}t^{\varepsilon_1}x_1 \cdots x_{r-1}t^{\varepsilon_r}w^{\rho_r}$ with $\varepsilon_1 = 1 = \varepsilon_r$. But much of Lemma 3 is needed for this and we have obtained additional information about automorphisms of G.

We now turn to the situation where we must impose the additional hypothesis that $n = 2$. This is done in order to deal with the situation where $F\phi \not\subseteq F$ (even allowing for inner automorphisms). We shall show that, in these circumstances, ϕ cannot be surjective.

Given $u_1, u_2, u_3 \not\in F$ we call (u_1, u_2, u_3) a *reduction triple* if in the process of reducing $u_1u_2u_3$ to normal form, the occurrences of t in u_2 are eliminated.

LEMMA 6. *Let $u_i \not\in F$, $i = 1, 2, \ldots, s$, suppose that $u_1u_2 \cdots u_s \sim F$. Then, for some i, (u_{i-1}, u_i, u_{i+1}) is a reduction triple–including the possibility that $i = 1$ or $i = s$ in which case $i - 1$ and $i + 1$ are to be interpreted modulo s.*

PROOF. If the conclusion were false the cyclically t-reduced form of $u_1u_2 \cdots u_s$ would not be free of occurrences of t. Q.E.D.

We begin with the situation in which $a_1\phi \not\in F$ and $a_2\phi \in F$. Firstly we require a rather technical lemma.

LEMMA 7. *Let $u \not\in F$ and $y \in F$ be such that $(u^\varepsilon, y^r u^\eta, y^s u^\nu)$, where $r, s \in \mathbf{Z}$ and $\varepsilon, \eta, \nu = \pm 1$ is a reduction triple. Then the following possibilities occur:*

(i) (a) $\varepsilon + \eta = 0$ *and* $y\theta = 1$ *or*

(b) $\eta + \nu = 0$ *and* $y\theta = 1$,

(ii) (a) $\varepsilon = \eta$, *and* $u^\varepsilon\theta = (y\theta)^{-r}$ *or*

(b) $\eta = \nu$, *and* $u^\eta\theta = (y\theta)^{-s}$,

(iii) $\varepsilon = \eta = \nu$ *and* $u^\varepsilon = u_0xu_0^{-1}y^{-r}$, *for some* $x \in F$ *and some* u_0.

(Recall that θ: $G \to L$ is the canonical epimorphism described at the outset.)

PROOF. It is clear that either $u^\varepsilon y^r u^\eta$ or $u^\eta y^s u^\nu$ is not in normal form.

Suppose the former occurs; if $\varepsilon + \eta = 0$ then $y^r \sim \langle w\rangle$ (in F). This gives $(y\theta)^r = 1$. Since w is not a proper power, K is torsion free (see [5]). Thus $y\theta = 1$ and (i)(a) holds.

Let $\varepsilon = \eta = 1$; if in fact $uy^ru \in F$ then analysis of the reduction of the uy^ru to normal form gives rise to a system of equations in the following way. Let $u = x_0t^{\kappa_1}x_1 \cdots t^{\kappa_p}x_p$. Then

$$x_py^rx_0 = w^{i_pl[\kappa_p]}, \qquad \kappa_p + \kappa_1 = 0,$$

$$x_{p-1}w^{i_pl[-\kappa_p]}x_1 = w^{i_{p-1}l[\kappa_{p-1}]}, \qquad \kappa_{p-1} + \kappa_2 = 0,$$

$$\vdots$$

$$x_1w^{i_2l[-\kappa_2]}x_{p-1} = w^{i_1l[\kappa_1]}, \qquad \kappa_1 + \kappa_p = 0,$$

where the group equations hold in F.

Now clearly $2(\kappa_1 + \cdots + \kappa_p) = 0$ and so $p = 2q$, $q \in \mathbf{Z}$. Then among the group equations there occurs

$$x_q w^{i_{q+1}l[-\kappa_{q+1}]} w_q = w^{i_q l[\kappa_q]}.$$

Clearly $(x_q\theta)^2 = 1$ and so $x_q\theta = 1$, since K is torsion free.

Let $u_0 = x_0 t^{\kappa_1} x_1 \cdots x_{q-1} t^{\kappa_q}$. The above equations imply that

$$x_q t^{\kappa_{q+1}} \cdots t^{\kappa_p} x_p = x_q w^{-i_{q+1}l[-\kappa_{q+1}]} u_0^{-1} y^{-r}$$

and hence that $u = u_0 \hat{x}_q u_0^{-1} y^{-r}$, where $\hat{x}_q = x_q w^{-i_{q+1}l[-\kappa_{q+1}]}$. But clearly $\hat{x}_q\theta = 1$ and so $u\theta = (y\theta)^{-r}$. The argument is similar when $\varepsilon = -1$ and we have (ii)(a).

So suppose that $u^\varepsilon v^r u^\eta \not\in F$. Then $u^\eta y^s u^\nu$ is not in normal form; if $\eta + \nu = 0$ we have (i)(b). So assume $\varepsilon = \eta = \nu$. If $u^\eta y^s u^\nu \in F$ we obtain (ii)(b). Otherwise, taking $\varepsilon = 1$, we obtain two systems of equations, viz.

$$x_p y^r x_0 = w^{i_p l[\kappa_p]}, \qquad \kappa_p + \kappa_1 = 0,$$

$$\vdots$$

$$x_c w^{i_{c+1}l[-\kappa_{c+1}]} x_{p-c} = w^{i_c l[\kappa_c]}, \qquad \kappa_c + \kappa_{p+1-c} = 0,$$

from reduction in $uy^r u$, and

$$x_p y^s x_0 = w^{k_p l[\kappa_p]}, \qquad \kappa_p + \kappa_1 = 0$$

$$\vdots$$

$$x_{p+2-c} w^{k_{p+3-c} l[\kappa_{p+3-c}]} x_{c-2} = w^{k_{p+2-c} l[\kappa_{p+2-c}]}, \qquad \kappa_{p+2-c} + \kappa_{c-1} = 0,$$

from reduction in $uy^s u$.

As before $2(\kappa_1 + \cdots + \kappa_p) = 0$ and $p = 2q$. It is easy to see that $c < q + 1$ or $c = q + 1$ or $p + 2 - c < q + 1$. If $c < q + 1$ we obtain $u\theta = (y\theta)^{-r}$ and if $p + 2 - c < q + 1$ we obtain $u\theta = (y\theta)^{-s}$, arguing as above.

Let $c = q + 1$; then we still have $u = u_0 \hat{x}_q u_0^{-1} y^{-r}$ by substituting appropriately. We cannot, however, conclude that $\hat{x}_q\theta = 1$; but we have (iii). A similar argument is given when $\varepsilon = -1$.

LEMMA 8. *Let $u \not\in F$, $y \in F$ be such that there is a word $z(a_1, a_2)$ such that $z(u, y) \sim F$. Then for any $g \in G$, $\langle u, y, g\rangle \neq G$.*

PROOF. Clearly we may assume that $z(a_1, a_2)$ is cyclically reduced. Rather less obviously we may assume that $z(a_1, a_2)$ is not a proper power. For suppose $z(a_1, a_2) = z_0(a_1, a_2)^p$. Then $z_0(u, y)^p \sim F$; we claim $z_0(u, y) \sim F$. Suppose not; then $z_0(u, y)$ can be expressed as $v_1 v_2 v_1^{-1}$ where v_2 is cyclically t-reduced, $v_2 \not\in F$. But then $z_0(u, y)^p = v_1 v_2^p v_1^{-1}$ and $v_2^p \nsim F$.

Let us write $z(a_1, a_2) = a_1^{\varepsilon_1} a_2^{r_1} a_1^{\varepsilon_2} a_2^{r_2} \cdots a_1^{\varepsilon_f} a_2^{r_f}$ where $\varepsilon_i = \pm 1$, $r_f \neq 0$, and $r_i \neq 0$ if $\varepsilon_i + \varepsilon_{i+1} = 0$. Then we have

$$u^{\varepsilon_1}y^{r_1}u^{\varepsilon_2}y^{r_2}\cdots u^{\varepsilon_f}y^{r_f}\sim F.$$

By Lemma 5, there is reduction triple $(u^{\varepsilon_{j-1}}, y^{r_{j-1}}u^{\varepsilon_j}, y^{r_j}u^{\varepsilon_{j+1}})$.

We claim that either $y\theta = 1$ or $u\theta \in \langle y\theta\rangle$. This is immediate if either (i) or (ii) of Lemma 7 occurs. So suppose that $\varepsilon_{j-1} = \varepsilon_j = \varepsilon_{j+1} = 1$ and $u = u_0xu_0^{-1}y^{-r}$ where $r = r_{j-1}$. (The argument when $\varepsilon_{j-1} = \varepsilon_j = \varepsilon_{j+1} = -1$ is virtually identical to what follows.) Substituting in $z(u,y)$ we obtain a word of the form

$$y^{s_0}u_0x^{\varepsilon_1}u_0^{-1}y^{s_1}\cdots u_0x^{\varepsilon_f}u_0^{-1}y^{s_f}$$

where $s_i = r_i - \frac{1}{2}(\varepsilon_i + \varepsilon_{i+1})r$, $i = 1, 2, \ldots, f$, $s_0 = 0$ or r according as $\varepsilon_1 = 1$ or -1, and $s_f = r_f - r$ or r_f according as $\varepsilon_f = 1$ or -1.

We firstly consider the possibility that $s_1 = s_2 = \cdots = s_{f-1} = 0$. Then $r_i = \frac{1}{2}(\varepsilon_i + \varepsilon_{i+1})r$, $i = 1, 2, \ldots, (f-1)$. This means $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_f$, for otherwise there exists i such that $\varepsilon_i + \varepsilon_{i+1} = 0$ and, hence, $r_i = 0$. Thus $r_1 = \cdots = r_{f-1} = \pm r$. Assume $\varepsilon_1 = 1$ so that $r_1 = r$. Then $z(u,y) = u_0x^fu_0^{-1}y^{r_f-r}$. Since $z(a_1, a_2)$ is not a proper power, $r_f \neq r$. The fact that $y_0x^fu_0^{-1}y^{r_f-r} \sim F$ means that either $x^f \sim \langle w\rangle$ or $y^{r_f-r} \sim \langle w\rangle$ in F. Hence $u\theta = 1$ or $v\theta = 1$. If $\varepsilon_1 = -1$ and $r_1 = -r$, then $z(u,y) = y^ru_0x^{-f}u_0^{-1}y^{r_f}$. Since $z(a_1, a_2)$ is not a proper power, $r_f \neq -r$. As above we obtain $u\theta = 1$ or $y\theta = 1$.

Now we may suppose that some $s_i \neq 0$, $i = 1, 2, \ldots, f-1$. This means that $z(u,y)$ is equal after free cancellation to a word of the form

$$y^{s_0}u_0x^{\lambda_1}u_0^{-1}y^{t_1}\cdots y^{t_{q-1}}u_0x^{\lambda_q}u_0^{-1}y^{s_f}$$

where $\lambda_i, t_i \in \mathbf{Z}$, $q \geqslant 2$, $t_i \neq 0$, $i = 1, 2, \ldots, f$, λ_1 has the same sign as ε_1, and λ_q the same sign as ε_f. Note that the initial and terminal powers of y are unchanged. Since $z(u,y) \sim F$, this word is not cyclically reduced. If $u_0x^{\lambda_i}u_0^{-1}$ or $u_0^{-1}y^{t_i}u_0$, for some i, is not in normal form we obtain respectively $x\theta = 1$ or $y\theta = 1$. The claim follows in this case. If the above word is in normal form then it follows that $u_0y^{s_f+s_0}u_0^{-1}$ is not in normal form. Then of course $y\theta = 1$ except when $s_f + s_0 = 0$ (this cannot immediately be ruled out). If $s_f + s_0 = 0$, then $u_0^{-1}x^{\lambda_q+\lambda_1}u_0$ is not in normal form. If $\lambda_q + \lambda_1 \neq 0$, this gives $x\theta = 1$ and we are all right. So assume $\lambda_q + \lambda_1 = 0$. Then of course ε_1 and ε_f are of opposite sign. If $\varepsilon_1 = 1$ and $\varepsilon_f = -1$, then $s_0 = 0$ and $s_f = r_f$ so that $r_f = 0$. This is contradictory. If $\varepsilon_1 = -1$ and $\varepsilon_f = 1$, then $s_0 = r$ and $s_f = r_f - r$ so again $r_f = 0$ and we have a contradiction.

We can now show that $\langle u, y, g\rangle \neq G$. If $\langle u, y, g\rangle = G$, then $\langle u\theta, y\theta, g\theta\rangle = L$. This means L is generated by at most two elements. By our usual argument, this contradicts the fact that w is not primitive.

PROPOSITION 9. *Let ϕ: $G \to G$ be an endomorphism such that $a\phi \notin F$ and $b\phi \in F$. Then ϕ is not surjective.*

PROOF. If ϕ is surjective then $G = \langle a\phi, b\phi, t\phi\rangle$. Since $w\phi \sim \langle w\rangle$ by Lemma 2 and $w\phi = w(a\phi, b\phi)$, Lemma 8 yields a contradiction.

For any $u \in G$, let $l_t(u)$ denote the number of occurrences of t in u.

PROPOSITION 10. *Let $u, v \in G$, $u, v \not\sim F$ be such that*
(1) *there exists a word $z(a_1, a_2)$ such that $z(u, v) \sim F$,*
(2) *there exists $g \in G$ such that $G = \langle u, v, g\rangle$.*
Then there exist $x, y \in F$ and $u_0 \in G$ such that $u = u_0 x u_0^{-1}$ and $v = u_0 y u_0^{-1}$.

PROOF. We proceed by induction on $l_t(u) + l_t(v)$. We therefore assume the proposition false and that among all pairs u and v satisfying the hypotheses, but not the conclusion, we have chosen a pair with $l_t(u) + l_t(v)$ minimal.

Suppose $u \sim F$, say $u = u_0 x u_0^{-1}$, $x \in F$. Then we have

$$u_0 x^{m_1} u_0^{-1} v^{n_1} \cdots u_0 x^{m_j} u_0^{-1} v^{n_j} \sim F$$

where $z(a_1, a_2) = a_1^{m_1} a_2^{n_1} \cdots a_1^{m_j} a_2^{n_j}$. Let $\hat{v} = u_0^{-1} v u_0$ and $\hat{g} = u_0^{-1} g u_0$. Then $G = \langle x, \hat{v}, \hat{g}\rangle$ and $z(x, \hat{v}) \sim F$. By Lemma 8, $\hat{v} \in F$, i.e. $v = u_0 y u_0^{-1}$, $y \in F$. Our minimality assumption is contradicted.

The argument when we assume $v \sim F$ is similar.

So suppose $u \nsim F$ and $v \nsim F$. Let $u = u_1 u_2 u_1^{-1}$ and $v = v_1 v_2 v_1^{-1}$ be in normal form with u_2 and v_2 cyclically t-reduced. Let $\hat{u} = v_1^{-1} u_1 u_2 u_1^{-1} v_1$ and $\hat{g} = v_1^{-1} g v_1$. Then $G = \langle \hat{u}, v_1, \hat{g}\rangle$ and $z(\hat{u}, v_2) \sim F$. If $\hat{u}$ (equivalently $u_1^{-1} v_1$) is not in normal form then the induction hypothesis gives $v_2 \sim F$ which is contradictory. So we may assume $\hat{u}$ is in normal form.

Case (i). Let $\hat{u}$ be cyclically t-reduced. Then, in practice, we have $u_1 = v_1 = 1$, i.e. $u = \hat{u}$ and $v = v_2$.

So we have

$$u^{m_1} v^{n_1} \cdots u^{m_j} v^{n_j} \sim F$$

where $z(a_1, a_2) = a_1^{m_1} a_2^{n_1} \cdots a_1^{m_j} a_2^{n_j}$ and u and v are cyclically t-reduced. By Lemma 6 we can find a reduction triple. The following are the only possibilities:

(1) $(u^\varepsilon, u^\varepsilon, v^\eta)$,
(2) $(u^\varepsilon, v^\eta, v^\eta)$,
(3) $(u^\varepsilon, v^\eta, u^\nu)$,
(4) $(v^\varepsilon, u^\eta, v^\nu)$,
where $\varepsilon, \eta, \nu = \pm 1$.

Suppose (1) occurs; then we can write $v^\eta = v_3 v_4$ where $u^\varepsilon v_3 \in F$. Thus $l_t(u^\varepsilon v) < l_t(v)$. Now $G = \langle u, v, g\rangle$; also we can regard $z(u, v)$ as $z(u, u^{-\varepsilon}(u^\varepsilon v))$ if $\eta = 1$ and as $z(u, (vu^{-\varepsilon})u^\varepsilon)$ if $\eta = -1$. In either case we can construct a word $z^*(a_1, a_2)$ such that $z^*(u, u^\varepsilon v^\eta) \sim F$. By the induction hypothesis $u \sim F$ which is contradictory. Clearly (2) is similar to (1).

Suppose (3) occurs. If $\varepsilon + \nu = 0$ then either $u^\varepsilon v^\eta$ or $v^\eta u^\nu$ is t-reduced–since

v is cyclically t-reduced. Suppose $v^\eta u^\nu$ is t-reduced. Then we can write $u^\epsilon = u_5 u_6$ where $u_6 v^\eta \in F$. Then $l_t(u^\epsilon v^\eta) < l_t(u)$ and $G = \langle u^\epsilon v^\eta, v, g\rangle$. Also we can transform $z(u, v)$ into $z^*(u^\epsilon v^\eta, v)$ and deduce that $v \sim F$ by the induction hypothesis. The alternative case is similar and so too is (4).

Case (ii). Suppose that $\hat{u}$ is not cyclically t-reduced. Write $u_3 = v_1^{-1}u_1$; thus $\hat{u} = u_3 u_2 u_3^{-1}$. We have

$$u_3 u_2^{m_1} u_3^{-1} v_2^{n_1} \cdots u_3 u_2^{m_j} u_3^{-1} v_2^{n_j} \sim F$$

and Lemma 6 ensures the existence of a reduction triple. The possibilities are:

(1) $(u_2^\epsilon, u_3^{-1}, v_2^\eta)$,
(2) $(v_2^\eta, v_2^\eta, u_3)$,
(3) $(u_3^{-1}, v_2^\eta, u_3)$,
(4) $(u_3^{-1}, v_2^\eta, v_2^\eta)$,
(5) $(v_2^\eta, u_3, u_2^\epsilon)$.

We examine (1) in detail. Here we must have $v_2^\eta = v_3 v_4$ where $u_3^{-1} v_3 \in F$; write $x = u_3^{-1} v_3$. Then $z(x^{-1} u_4 x, (v_4 v_3)^\eta) \sim F$ and $G = \langle x^{-1} u_4 x, (v_4 v_3)^\eta, v_3^{-1} \hat{g} v_3\rangle$. Since $l_t(x^{-1} u_4 x) < l_t(\hat{u})$ we obtain $(v_4 v_3)^\eta \sim F$ and hence $v_2 \sim F$.

The remaining cases are dealt with in a similar manner and the proof is complete. Q.E.D.

We come finally to the proof of our main theorem.

Proof of the Theorem. Let ϕ be a surjective endomorphism. By Proposition 9, either $\langle a_1\phi, a_2\phi\rangle \subseteq F$ or $a_1\phi \not\subseteq F$ and $a_2\phi \not\subseteq F$. If $\langle a_1\phi, a_2\phi\rangle \subseteq F$ then by Proposition 5, ϕ is an automorphism. So let $a_1\phi \not\subseteq F$ and $a_2\phi \not\subseteq F$. By Lemma 2 $w\phi \sim \langle w\rangle$. We may apply Proposition 10 with $z = w$, $u = a_1\phi$, $v = a_2\phi$. Thus there exists an inner automorphism ψ such that $\langle a_1\phi\psi, a_2\phi\psi\rangle \subseteq F$. By Proposition 5, $\phi\psi$ is an automorphism and hence ϕ is an automorphism. Q.E.D.

Our results enable us to say something about the automorphism group of G. Let $S = \{\mu \in \text{Aut } F\colon\ w\mu = w^\epsilon, \epsilon = \pm 1\}$. S is the group of all automorphisms μ of F such that $\langle w\mu\rangle = \langle w\rangle$.

Lemma A. *Let $\mu \in S$. Then the mapping $\mu^*\colon G \to G$ defined by*

$$\mu^*\colon \begin{cases} x \to x\mu, & x \in F, \\ t \to t \end{cases}$$

is an automorphism of G.

Proof. It is easy to see that $(t^{-1} w^l t)\mu^* = (w^m)\mu^*$. Since μ^* is clearly surjective it is an automorphism. Q.E.D.

Lemma B. S is embedded in Aut G.

PROOF. The map sending $\mu \to \mu^*$ as above is easily seen to be an embedding. Q.E.D.

LEMMA C. *Let* $\phi \in \text{Aut}\, G$ *be such that* $F\phi \subseteq F$. *Then there exists* $x \in F$ *such that*

(i) $x^{-1}(w\phi)x = w^{\varepsilon}, \qquad \varepsilon = \pm 1,$

(ii) $x^{-1}(t\phi)x = tw^{p}.$

PROOF. This is proved by an easy adaptation of the proof of Proposition 5. Q.E.D.

LEMMA D. *Let* $\phi \in \text{Aut}\, G$ *be such that* $F\phi \subseteq F$ *and* $w\phi = w^{\varepsilon}$, $\varepsilon = \pm 1$. *Then* $F\phi = F$.

PROOF. Suppose not; then, say, $a_1 \not\in F\phi$. Since ϕ is surjective and $t\phi = w^{p_0}tw^{p_1}$ it is clear that $a_1 \in \langle F\phi, w, t\rangle$. We claim that $a_1 \in \langle F\phi, w\rangle$.

Let $a_1 = y_0 t^{\eta_1} y_1 \cdots t^{\eta_s} y_s$ where $y_i \in \langle F\phi, w\rangle$, $i = 0, 1, 2, \ldots, s$. If $s = 0$, then there is nothing to prove. If $s > 0$, then the right-hand side is not t-reduced. So for some i, $t^{\eta_i} y_i t^{\eta_{i+1}} \in \langle w^{l[-\eta_i]}\rangle$. By induction, $a_1 \in \langle F\phi, w\rangle$.

However, $w = w^{\varepsilon}\phi$, $\varepsilon = \pm 1$, and thus $w \in F\phi$. This is contradictory. Q.E.D.

Write Inn G for the group of inner automorphisms of G.

PROPOSITION E. *Let* $\phi \in \text{Aut}\, G$. *Then there exists* $\psi \in \text{Inn}\, G$ *such that*

(i) $F\phi\psi = F$,

(ii) $w\phi\psi = w^{\varepsilon}$, $\varepsilon = \pm 1$,

(iii) $t\phi\psi = \text{-}w^{p}$, *some* $p \in \mathbf{Z}$.

PROOF. This is immediate from Proposition 9 and the Lemmas just above. Q.E.D.

We can now describe Aut G.

Let S^* be the copy of S embedded in Aut G. Let σ: $G \to G$ be the automorphism of G defined by

$$\sigma^*: \begin{array}{ll} x \to x, & x \in F, \\ t \to tw. & \end{array}$$

Then Aut $G = \langle S^*, \sigma, \text{Inn}\, G\rangle$. For let $\phi \in \text{Aut}\, G$; let ψ be such that (i), (ii) and (iii) of Proposition E are satisfied. Let μ be the restriction to F of $\phi\psi$. Then $\phi\psi = \mu^*\sigma^p$.

We note the following facts:

(a) $\langle\sigma\rangle$ is infinite cyclic.

(b) $\langle S^*, \sigma\rangle = S^* \times \langle\sigma\rangle$, the direct product of S^* and $\langle\sigma\rangle$.

(c) Inn $G \cong G$ since G has trivial centre.

(d) $\langle S^*, \sigma\rangle \cap \operatorname{Inn} G = \langle \beta^l\rangle$ where β is the inner automorphism of G corresponding to w.

To see (d), let $\phi \in \langle S^*, \sigma\rangle \cap \operatorname{Inn} G$. If $\phi = \mu^*\sigma^p$, then for some $v \in G$ we have $v^{-1}tv = tw^p$ and also $v^{-1}wv = w^\varepsilon$, $\varepsilon = \pm 1$. If $|l|, |m| \neq 1$, the latter immediately yields $v \in \langle w\rangle$ and so $\phi \in \langle \beta\rangle$. Then the first of the two equalities yields $v \in \langle w^l\rangle$. Since $\beta^l = (\gamma^*)^l\sigma^{l-m}$ where γ is the inner automorphism of F corresponding to w, we have the desired conclusion.

Suppose that $l = 1$; then the equation $v^{-1}wv = w^\varepsilon$ yields only the fact v is of the form $w^{q_0}t^{\eta_1}w^{q_1}\cdots t^{\eta_s}w^{q_s}$ where $\sum_{i=1}^k \eta_i > 0$, $i = 1, 2, \ldots, s$ (and possibly $s = 0$). However, a simple induction argument shows that this is inconsistent with $v^{-1}tv = tw^p$ unless $s = 0$. Then we can argue as before.

In conclusion we remark that our description of Aut G is only a relative one since it depends on knowledge of S. J. McCool has proved in [6] that S is always finitely presented. Thus we know that Aut G is always finitely generated. It remains to be seen whether or not Aut G is also finitely related.

References

1. G. Baumslag, *Residually finite one-relator groups*, Bull. Amer. Math. Soc. **73** (1967), 618–620. MR **35** #2953.

2. G. Baumslag and D. Solitar, *Some two-generator, one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68** (1962), 199–201. MR **26** #204.

3. G. Baumslag and A. Steinberg, *Residual nilpotence and relations in free groups*, Bull. Amer. Math. Soc. **70** (1964), 283–284. MR **28** #1229.

4. I. M. S. Dey and H. Neumann, *The Hopf property of free products*, Math. Z. **117** (1970), 325–339. MR **43** #2099.

5. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, Interscience, New York, 1966. MR **34** #7617.

6. J. McCool, *Some finitely presented subgroups of the automorphism group of a free group*, J. Algebra **35** (1975), 205–213.

7. S. Meskin, *Nonresidually finite one-relator groups*, Trans. Amer. Math. Soc. **164** (1972), 105–114. MR **44** #2807.

8. C. F. Miller III, *On group-theoretic decision problems and their classification*, Ann. of Math. Studies, no. 68, Princeton Univ. Press, Princeton N. J.; Univ. of Tokyo Press, Tokyo, 1971. MR **46** #9147.

Department of Pure Mathematics, Queen Mary College, London, England